Concludes Calculation of Fluctuation Spectrum.

ii.) Test Particle Model - Fluctuation Spectrum

→ Basic ideas:

- thermal equilibrium of (stable) plasma is balance of:

→ Cerenkov emission of plasma waves by discrete particles (i.e. wake)

→ absorption of waves by Landau damping.

- Key ideas:

→ weak fluctuations - linear trajectories (unperturbed orbits)

→ uncorrelated emission (of / by) test particles

→ each particle a "double agent":

1) → discrete emitter

2) → part of the phase space fluid which absorbs the emission from other test particles.

n.b.: plasma ~ pea soup (husks + soup)

discrete particles → husks + broth → phase space fluid

"every pea in the pea soup is part of the soup to other peas".

Consider electron plasma, with stationary ions so:

For test particle potential, extend Debye calculation:

$$\delta f = \underbrace{f^c}_{\substack{\text{coherent}\\ \text{Vlasov}\\ \text{response}}} + \underbrace{\tilde{f}}_{\substack{\text{discreteness}\\ \text{source}}}$$

$$\delta f = \underbrace{\frac{\frac{|e|}{m}\tilde{E}_{k,\omega}\,\partial\langle f\rangle/\partial v}{-i(\omega - kv)}}_{\text{coherent response}} + \underbrace{|e|\,\delta(x - x(t))\,\delta(v - v(t))}_{\text{discrete source}}$$

$$\nabla^2\phi = 4\pi n_0 |e| \int dv\, \delta f$$
$$= 4\pi n_0 |e| \int dv\, f^0 + 4\pi n_0 |e| \int dv\, \tilde{f}$$

so

$$\epsilon(k,\omega)\, \hat{\phi}_{k,\omega} = \frac{4\pi n_0 |e|}{k^2} \int dv\, \tilde{f}_{k,\omega}$$

Now, using u.p.o. :

$$\int \tilde{f}_k = \int dx\, e^{-ikx} |e|\, \delta(x - x(t))$$
$$x(t) = \cancel{x_0} + vt$$

so

$$\underline{\underline{\epsilon}}(k,t)\, \hat{\phi}(\underline{k},t) = \frac{4\pi n_0 |e|}{k^2} e^{-ikvt}$$

$\Rightarrow$

$$\phi_k(t) = \underline{\underline{\epsilon}}^{-1}(k,t) \frac{4\pi n_0 |e|}{k^2} e^{-ikvt}$$
$$+ \phi_k^{homog} e^{-i\omega_k t}$$

$\rightarrow$ driven solution (discreteness of damping)

homog. solution

$\omega_k = \omega_r(k) + i\,\omega_i(k)$

so time asymptotically:

- $\omega_i(k) < 0 \Rightarrow$ homogeneous collective response damped.

- only driven (by discreteness) solutions persist

but:

- $\omega_i \lesssim 0$, may need wait long time

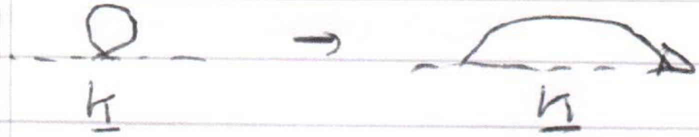
- for sufficient source strength, fluctuations in system may grow to nonlinearity before damping occurs.

- if unstable modes require ultimate nonlinear damping to balance noise (i.e. $\epsilon_{IM} = \epsilon_{IM}(k, \omega, \langle\tilde{\phi}^2\rangle)$)

  $\rightarrow$ "noise" = thermal + nonlinear, then

  $\Rightarrow$ transfer, as well as emission and absorption, occurs.

then



→ Have:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2} \right)^2 \int dv_1 \int dv_2 \, \frac{\langle \tilde{f}(1)\, \tilde{f}(2) \rangle_{k,\omega}}{|\epsilon(k,\omega)|^2}$$

∴ all context:

- Coulomb factor
- discreteness source $\langle \tilde{f}\tilde{f} \rangle_{k,\omega}$
- $\epsilon(k,\omega)$ collective response

→ Noise

$$\tilde{f} = \frac{1}{n} \sum_{i=1}^{N} \delta(x - x_i(t))\, \delta(v - v_i(t))$$

→ u.p.o. $\begin{cases} x_i(t) = x_{i,0} + v_i t \\ v_i(t) = v, \ \text{const} \end{cases}$

$$\langle \ \rangle = n \int dx_i \int dv_i \, \langle f(v_i) \rangle$$

↳ ~ Maxwellian

↓

avg. over eqbrm distrib. of discr. uncorrelated test particles

"uncorrelated" → dilute, $k_B T \gg e^2/\bar{r}$

→ $1/n\lambda_D^3 < 1$

so

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle = \Big\langle n \int dx_i \int dv_i \Big(\frac{1}{n} \sum_{i=1}^{N} \delta(x_1 - x_i(t))\, \delta(v_1 - v_i(t)) \Big) \Big(\frac{1}{n} \sum_{j=1}^{N} \delta(v_2 - v_j(t))\, \delta(x_2 - x_j(t)) \Big) \Big\rangle$$

only $\neq 0$ if arguments interchangeable !

$$= \int dx_i \int dv_i \, \frac{\langle f \rangle}{n} \sum_{i,j}^{N} \delta(x_1 - x_2)\, \delta(x_i - x_j)\, \delta(v_1 - v_2) \ast \delta(v_i - v_j)$$

$$= \frac{\langle f \rangle}{n} \delta(x_1 - x_2)\, \delta(v_1 - v_2)$$

so, discreteness correlation function:

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle = \frac{\langle f\rangle}{n}\,\delta(\underline{x}_1-\underline{x}_2)\,\delta(\underline{v}_1-\underline{v}_2)$$

i.e. - no width, no range
- particles only correlated if same particle.

Now, need:

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega}$$

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_k = \int e^{-ik(x_2-x_1)}\langle \tilde{f}(1)\tilde{f}(2)\rangle$$

and time transform $\rightarrow$ time history (u.p.o)

$$\langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega} \equiv \int_0^{\infty} d\tau\, e^{i\omega\tau}\, U(2,\tau)\int e^{-ik(x_2-x_1)}\,dx_-\,\langle \tilde{f}(1)\tilde{f}(2)\rangle$$

$$+\int_{-\infty}^{0} d\tau\, e^{i\omega\tau}\, U(1,-\tau)\int e^{-ik(x_2-x_1)}\langle \tilde{f}(1)\tilde{f}(2)\rangle\, dx_-$$

$U \rightarrow$ operator pushing particle along u.p.o.

①: $x \to x + v\tau$

②: $x_2 \to x_2 + v_2\tau$

$$② = \int_0^\infty d\tau \int e^{-ik(x_2 - x_1)} e^{i(\omega - kv_2)\tau} \langle \tilde{f}(1)\tilde{f}(2)\rangle dx_-$$

$$= \int_0^\infty d\tau \, e^{i(\omega - kv_2)\tau} \langle \tilde{f}(1)\tilde{f}(2)\rangle_k$$

$$= \frac{-1}{i(\omega - kv_2)} \langle \tilde{f}\tilde{f}\rangle_{\underline{k}}$$

$$= \frac{i}{(\omega - kv_2)} \langle \tilde{f}\tilde{f}\rangle_{\underline{k}}$$

seek $\underline{re}$)

$$= \pi\delta(\omega - kv_2) \langle \tilde{f}\tilde{f}(k)\rangle$$

Similarly,

$$① = \pi\delta(\omega - kv_1)\langle \tilde{f}\tilde{f}\rangle_k$$

and seek:

$$\int dv_1 \int dv_2 \langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega}$$

so

$$\langle \tilde{f}(1)\,\tilde{f}(2)\rangle = \frac{\langle f\rangle}{n}\,\delta(x_-)\,\delta(v_-)$$

$$\langle \tilde{f}\tilde{f}\rangle_k = \frac{\langle f\rangle}{n}\,\delta(v_-)$$

$$\int dv_1 \int dv_2 = \int dv_+ \int dv_-$$

$v_- \to$ relative
$v_+ \to$ CM

so

$$\boxed{\int dv_-\ \langle \tilde{f}(1)\,\tilde{f}(2)\rangle_{k,\omega} = 2\pi\,\delta(\omega - kv)\,\frac{\langle f\rangle}{n}}$$

and

$$\left\langle \frac{\tilde{n}}{n_0}\frac{\tilde{n}}{n_0}\right\rangle_{k,\omega} = \int dv\,\frac{\langle f\rangle}{n}\,2\pi\,\delta(\omega - kv)$$

$$\equiv C(k,\omega)$$

emission correlator

$\hookrightarrow v_{the}$ extracted.

$$C(k,\omega) = \frac{2\pi}{n|k|v_{the}} \langle \bar{f}(\omega/k v_{the}) \rangle$$

$$= \frac{2\pi}{n} \tau_{str} \langle \bar{f}(\omega/k v_{the}) \rangle$$

$\hookrightarrow$ streaming time

discreteness noise has Maxwellian Doppler Spectrum (obvious)

so, thermal equilibrium spectrum:

$$\boxed{\langle \phi^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 |e|}{k^2} \right)^2 \frac{1}{n_0} \frac{2\pi}{|k| v_{the}} \frac{\langle \bar{f}(\omega/k v_{the}) \rangle}{|\epsilon(k,\omega)|^2}}$$

so, spectrum set by:

- equilibrium particle emission distribution $\sim e^{-\omega^2/k^2 v_{the}^2}$

- collective resonances, i.e

$\omega \geq k v_{the}$ $\quad \epsilon \cong 1 - \omega_p^2/\omega^2 + i\epsilon_{IM}$

$\omega < k v_{th}$ $\quad \epsilon \simeq 1 + 1/k^2\lambda_D^2 + i\epsilon_{IM}$.

- Coulomb factor (screening modified)
- γ_{str} (from time transform)

so

- collective response strongest at wave resonance ($kv \sim \omega_{pe} \sim \omega$)

$\Rightarrow$ expect peak in frequency spectrum

(n.b. emission distribution hits wave resonance $kv_{th} \sim \omega_{pe} \Rightarrow k \sim \lambda_D^{-1}$, for scale)

Limiting behavior:

- for $\omega > \omega_p$, noise source decouples from collective dynamics (i.e. $\epsilon \to 1$ as $\omega \gg \omega_{pe}$);

$$\langle \hat{\phi}^2 \rangle_{k,\omega} \cong n_0 \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{2\pi}{|k| v_{the}} e^{-\omega^2/k^2 v_{the}^2}$$

- for $\omega < \omega_p$, low frequency noise $\cong$ static $\Rightarrow$ screened by plasma so

$$\langle \phi^2 \rangle_{k,\omega} \cong n_0 \left(4\pi |e|\right)^2 \frac{2\pi}{|k| v_{The}} \frac{e^{-\omega^2/k^2 v_{th}^2}}{\left(k^2 + 1/\lambda_D^2\right)^2}$$

$\uparrow$ abs k^2

so

Can write Electric Field Spectrum:

$$\frac{\langle E^2 \rangle_{k,\omega}}{8\pi} = \frac{4\pi^2 n_0 |e|^2}{k^2 |k|} \frac{\underbrace{F(\omega/k v_{The})}_{\langle F \rangle / v_{The} \sqrt{2\pi}}}{\left[\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 + \left(\frac{\pi \omega_{pe} F'}{|k| k}\right)^2\right]}$$

$\hookrightarrow$ re-absorbs.

$$\left(F' = dF/du \atop u = \omega/k\right) \qquad |\epsilon_r|^2 + |\epsilon_{IM}|^2$$

i.e. Thermal Eqbrm. Spectrum

$$\frac{\langle E^2 \rangle_{k,\omega}}{8\pi} = \frac{4\pi^2 n_0 |e|^2}{k^2 |k|} \left(F / |\epsilon_r|^2 + |\epsilon_{IM}|^2\right)$$

Now, to make contact with usual expectations of "$k_b T/2$ per d-o-f".

$$W_k = \int \frac{d\omega}{2\pi} \langle E^2 \rangle_{k,\omega} / 8\pi$$

field energy per mode

Useful trick; Pole Approximation:

$$1/|\epsilon|^2 = 1 \Big/ \Big[\underbrace{(\omega-\omega_k)^2 \left|\left(\frac{\partial\epsilon}{\partial\omega}\right)\right|^2}_{\text{real frequency sets location}} + \underbrace{|\epsilon_{IM}|^2}_{\text{width}} \Big]$$

$$\cong \frac{1}{|\epsilon_{IM}|} \left\{ \frac{|\epsilon_{IM}|}{(\omega-\omega_k)^2 \left|\frac{\partial\epsilon}{\partial\omega}\right|^2 + |\epsilon_{IM}|^2} \right\}.$$

$$\cong \frac{1}{|\epsilon_{IM}|} \left[\left|\frac{\partial\epsilon}{\partial\omega}\right|_{\omega_k} \right]^{-1} \underbrace{\pi\, \delta(\omega-\omega_k)}_{\text{evals on collective resonance}}$$

so, pole approximation:

$$1/|\epsilon|^2 = \frac{\pi\,\delta(\omega-\omega_k)}{|\epsilon_{Im}(k,\omega_k)|\,|\partial\epsilon_r/\partial\omega|_{\omega_k}}$$

so integrating in pole approx:

$$W_k = \frac{m_e\,\omega_{pe}}{2|k|}\; F\Big/|F'|$$

$$= \frac{m_e\,\omega_{pe}}{2|k|}\; F\Big/\frac{\omega_{pe}}{|k|\,v_{Te}^2}F = T/2$$

in accord with "$T/2$ per d-o-f" intuition.

$\Rightarrow$ if $k\lambda_D \gg 1$, $\epsilon \Rightarrow 1 + 1/k^2\lambda_D^2$

no collective resonance

$$W_k \cong \frac{T}{2}\, 1/k^2\lambda_D^2 \rightarrow$$ strong cut-off beyond λ_D.

so, for total energy density: (3D)

$$\left\langle E^2/8\pi \right\rangle = \int d\underline{k}\, W_{\underline{k}}$$

$$\sim \left(\frac{k_B T}{2}\right) k_{max}^3$$

$$\sim \frac{n\, k_B T}{2} \frac{1}{n\lambda_D^3}$$

$$\sim (PKED)/\underbrace{n\lambda_D^3}_{\#\text{ in Debye sphere}}$$

consistent with idea of diluteness:

$$\boxed{\begin{array}{l} (FED) \sim (PKED)/n\lambda_D^3 \\ 1/n\lambda_D^3 \sim \text{diluteness/discreteness factor} \end{array}}$$

→ To connect formally, to $\boxed{\text{Fluctuation-dissipation theorem}}$:

notes $\epsilon_{IM} = -\frac{\omega_p^2 \pi}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k}$

$$= \frac{2\pi\omega}{k^2 v_{Te}^2} \frac{\omega_p^2}{|k| v_{Te}} \langle \bar{f}(\omega/k) \rangle$$, for Maxwellian $\langle f \rangle$

so $\langle \bar{f}(\omega/k) \rangle = k^2 v_{Te}^2 |k| v_{Te}\, \epsilon_{IM} \Big/ 2\pi\omega\, \omega_p^2$

have:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{2\pi n}{|k| v_{Te}} \left(\frac{4\pi|e|}{k^2}\right)^2 \frac{\langle \bar{f}(\omega/k) \rangle}{|\epsilon(k,\omega)|^2}$$

so, plugging in:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{8\pi T}{k^2 \omega} \frac{\operatorname{Im}\epsilon}{|\epsilon|^2}$$

$\langle \hat{u} \rangle \sim \delta T$ form

and

$$\boxed{\frac{\langle \hat{E}^2 \rangle_{k,\omega}}{8\pi} = \frac{T}{\omega} \frac{\operatorname{Im}\epsilon}{|\epsilon|^2}}$$

⟺ {Fluctuation-Dissipation Theorem}

→ relates thermal fluctuations to dissipation in collective modes ($\operatorname{Im}\epsilon$)

(restates form of spectrum)

→ obviously consistent (by construction), with physical ~~model~~ picture.

5) Some general comments:

→ Key element of T.P.M. is $\left\{ \begin{array}{l} \underline{\text{causality}} \\ \text{use of} \end{array} \right.$
linear $f^c_{k,\omega}$ or, equivalently, unperturbed orbit

- This assumes small fluctuation levels, so stochastic deflection is 'weak'

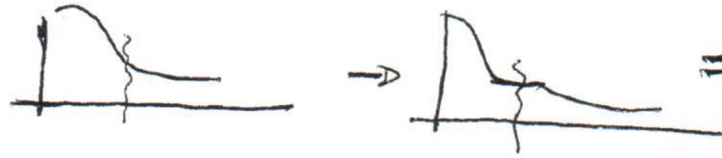
i.e. $\underline{X}(t) = \underline{X}(0) + \underline{V}t + \underbrace{\delta(\underline{X}(t))}_{\text{deflection}}$

How weak? ↔ take care $\tau_{ac} < \tau_T$ condition

<u>but</u>

- $\langle \tilde{\phi}^2 \rangle_k \sim (\ )\ \dfrac{F(\omega_{pe}/k)}{|F'(\omega_{pe}/k)|}$

∴ Fluctuations diverge as $F' \to 0$, from below

i.e.  ⇒ "Critical opalescence"

Note $F' > 0$ <u>not</u> necessary ↔ theory fails for <u>stable</u> <u>plasma</u> ...., approaching marginality.

→ As fluctuations grow, linearizations fail

∴ must renormalize:

→ particle propagator

$$i/\omega - kv \rightarrow i/\omega - kv + \underbrace{\Sigma}_{\text{self energy, decorrelation rate}}$$

→ mode propagator / response

$$1/\epsilon \rightarrow 1\Big/\left[\underbrace{(\omega-(\omega_k+\delta\omega_k))\frac{\partial\epsilon}{\partial\omega}}_{\substack{\text{nonlinear}\\\text{frequency}\\\text{shift}}} + i\underbrace{(\epsilon_{IM}^{L}+\epsilon_{IM}^{NL})}_{\substack{\text{nonlinear}\\\text{dissipation}}}\right]$$

(w-w interaction)
(w-p interaction)
⇒ γ^{NL}

(recall NL oscillator, driven!)

Calculating all this is aim of plasma turbulence theory ....